

# The dilute Temperley-Lieb $O(n = 1)$ loop model on a semi infinite strip: the sum rule

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## Abstract

This is the second part of our study of the ground state eigenvector of the transfer matrix of the dilute Temperley-Lieb loop model with the loop weight  $n = 1$  on a semi infinite strip of width  $L$  [12]. We focus here on the computation of the normalization (otherwise called the sum rule)  $Z_L$  of the ground state eigenvector, which is also the partition function of the critical site percolation model. The normalization  $Z_L$  is a symmetric polynomial in the inhomogeneities of the lattice  $z_1, \dots, z_L$ . This polynomial satisfies several recurrence relations which we solve independently in terms of Jacobi-Trudi like determinants. Thus we provide a few determinantal expressions for the normalization  $Z_L$ .

## 1 Introduction

The inhomogeneous loop models on the two dimensional square lattice on semi-infinite domains with different boundary conditions (cylinder, strip) were actively studied in the last decade. This concerns, in particular, the cases when the loop weight  $n = 1$ , and we will assume that everywhere below. Most famous examples of these models are: the Temperley-Lieb (TL) loop model [1, 23, 3, 19, 18, 9, 8, 15, 5, 4, 25], the Brauer loop model (BL) [9, 17, 8, 22] and recently the dilute Temperley-Lieb (dTL) loop model [7, 12, 6]. These models have many connections to combinatorics, critical percolation, geometric representation theory, etc. These connections are discussed in more details in the given references.

Usually we are interested in the computation of the ground state eigenvector of the transfer matrix, its normalization and correlation functions. Thanks to the fact that the ground state has polynomial entries it was possible to develop a procedure to compute these entries using some  $q$ -difference equations (loosely called the qKZ equations) and certain recurrence relations [9]. We have done this computation for the dTL model with open

boundary conditions in [12]. In the present work we solve the problem of the computation of the normalization  $Z_L$  of the ground state  $\Psi_L$  of the dTL model with open boundaries.

Like in the other loop models (TL and BL)  $Z_L$  is a symmetric polynomial and it obeys certain recurrence relations. In fact, it obeys two recurrence relations which are related to the fact that the  $R$ -matrix of the dTL model factorizes into two operators when its spectral parameter is equal to special values<sup>1</sup>. In particular, one such factorization gives rise to an operator which maps  $\Psi_L$  to  $\Psi_{L-1}$ . The normalization then satisfies the recurrence:

$$Z_L(z_1, \dots, z_{L-1} = z_{L-1}\omega, z_L = z_{L-1}/\omega) = F(z_1, \dots, z_{L-2}|z_{L-1})Z_{L-1}(z_1, \dots, z_{L-1}). \quad (1)$$

This recurrence relation fixes  $Z_L$  completely once we fix its initial condition. The computation of the polynomial  $F$  is a result of our previous work [12], we will specify it later. The second recurrence relation, which is unrelated to (1) has the form:

$$Z_L(z_1, \dots, z_{L-1} = z_{L-1}, z_L = -z_{L-1}) = P(z_1, \dots, z_{L-2}|z_{L-1})Z_{N-2}(z_1, \dots, z_{L-2}), \quad (2)$$

where the polynomial  $P$  will be given later. As in (1) this recurrence relation has a unique solution given the initial condition. This recurrence also comes from a factorization property of the  $R$ -matrix. Both recurrence relations (1) and (2) can be regarded as being "inherited" from the  $R$  matrix of the  $U_q(A_2^{(2)})$  integrable model. This will be discussed elsewhere in more detail [11].

These recurrences, with different functions  $F$  and  $P$ , say  $F^p$  and  $P^p$  (where the superscript  $p$  will always mean periodic), appear in the dTL model with periodic boundary conditions [7]. Let us call  $Z_L^p$  the normalization of the ground state vector of the periodic transfer matrix of the dTL model, then

$$Z_L^p(z_1, \dots, z_{L-1} = z_{L-1}\omega, z_L = z_{L-1}/\omega) = F^p(z_1, \dots, z_{L-2}|z_{L-1})Z_{L-1}^p(z_1, \dots, z_{L-1}). \quad (3)$$

$$Z_L^p(z_1, \dots, z_{L-1} = z_{L-1}, z_L = -z_{L-1}) = P^p(z_1, \dots, z_{L-2}|z_{L-1})Z_{N-2}^p(z_1, \dots, z_{L-2}), \quad (4)$$

In this case (3) is solved by a determinant of the elementary symmetric polynomials [7] (or homogeneous symmetric polynomials), which is, in fact, a skew Schur function. We are going to use this solution to find a determinantal expression solving (1). We will also show how to compute  $Z_L$  and  $Z_L^p$  using the second recurrences (2) and (4), respectively. The second recurrences also give determinant expressions for  $Z_L$  and  $Z_L^p$ . These expressions involve different symmetric polynomials. The two solutions, obtained independently, must be related by some transformation which is unknown to us.

A nice byproduct observation in the course of our computations was realizing that the polynomial  $P^p(z_1, \dots, z_L|\zeta)$  is the Baxter's  $Q$ -function [2] (the ground state eigenvalue of the  $Q$ -operator) for the corresponding spin chain, which is the already mentioned  $U_q(A_2^{(2)})$  integrable model (Izergin-Korepin model [16]). This means that the roots of this polynomial, regarded as a polynomial in  $\zeta$ , are the Bethe roots of the ground state eigenvector in the algebraic Bethe ansatz of the  $U_q(A_2^{(2)})$  vertex model. In particular, it allowed us to reproduce small systems results of the computation of the ground state components that we obtained

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<sup>1</sup>This fact is related to the quasi-triangularity of the Hopf algebra  $U_q(A_2^{(2)})$ . The algebra  $U_q(A_2^{(2)})$  defines the Izergin-Korepin vertex model, which is related to the dTL model by a certain basis transformation.

alternatively using the qKZ equations in our first work on the subject. This is possible since the ground states of the loop model and the one of the vertex model are related by a linear transformation [21]. For small systems ( $L \leq 5$ ) it is possible to match them completely. The details about the discussion of this paragraph will appear in [11].

The outline of the paper is as follows. We will start by introducing the dTL model at  $n = 1$  in the second section. For a more detailed introduction we refer to [12]. In the third section we will show how to solve the recurrence (1) for  $Z_L$  using the known solution  $Z_L^p$  of (3). In the fourth section we show how to solve (4) and (2). The conclusion is given in the fifth section.

## 2 The model

The dTL loop model is defined on a square lattice by decorating the faces of the lattice with one of the nine plaquettes (fig.(1)) in such a way that all loops in the bulk are continuous, they may end on the boundaries or form closed cycles in the bulk. We consider the model on a semi-infinite strip which is finite in the horizontal direction and infinite in the vertical. If we identify the two vertical boundaries of the strip, then the boundary conditions are called periodic. If we forbid loops to end at the vertical boundaries then we need to include two boundary plaquettes (the third and fifth on fig.(2)). This boundary conditions are called closed or reflecting. If we allow loops to end at the vertical boundaries, as on the example fig.(3), then it gives rise to open boundary conditions, and we need to consider three more boundary plaquettes along with the two of the reflecting case. All five boundary plaquettes are presented on fig.(2). The dTL model with open boundary conditions is the one we study here. We also shortly discuss and present a result for the periodic dTL model.

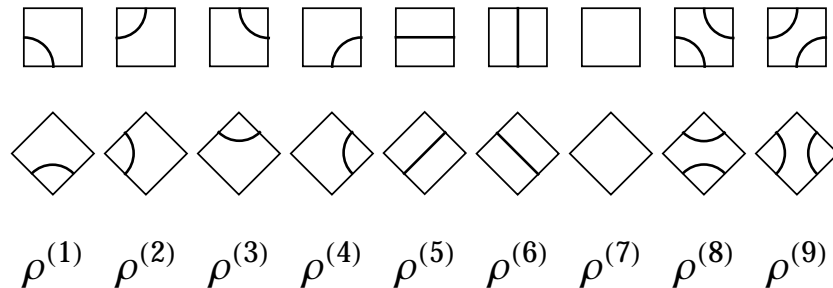


Figure 1: The bulk plaquettes (first row). The graphical representation of the nine operators (second row) acting on the link patterns (see fig.(6)). Graphically these are the 45 degrees tilted versions of the bulk plaquettes, they are called  $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(9)}$  respectively.

The operators  $\rho^{(i)}$  as well as  $\kappa_l^{(i)}$  and  $\kappa_r^{(i)}$  naturally act in the space of link patterns  $LP_L$ . The space  $LP_L$  is spanned by all possible connectivities of  $L + 2$  vertices on a straight horizontal line with certain restrictions. The first and the last vertex are called the boundary vertices, while the vertices in between are the bulk vertices. A bulk vertex can be disconnected from any other vertex (thus called unoccupied) or connected (occupied) only once to another bulk or boundary vertex. A boundary vertex can be disconnected from the rest vertices or connected any number of times to distinct bulk vertices (not the other boundary

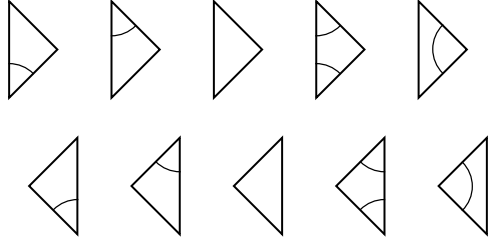


Figure 2: The left (top row) and the right (bottom row) boundary plaquettes. The corresponding five left boundary operators will be called  $\kappa_l^{(1)}, \dots, \kappa_l^{(5)}$  and the right boundary operators  $\kappa_r^{(1)}, \dots, \kappa_r^{(5)}$ , respectively.

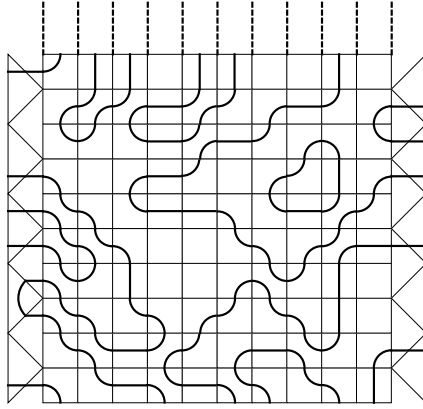


Figure 3: A typical configuration of the dilute  $O(n)$  loop model on a fragment of the semi-infinite strip.

vertex). We also require that there are no crossings in the connectivity. For  $L = 3$  all possible connectivities, or the basis elements of  $LP_3$ , are depicted on fig.(4). Every configuration

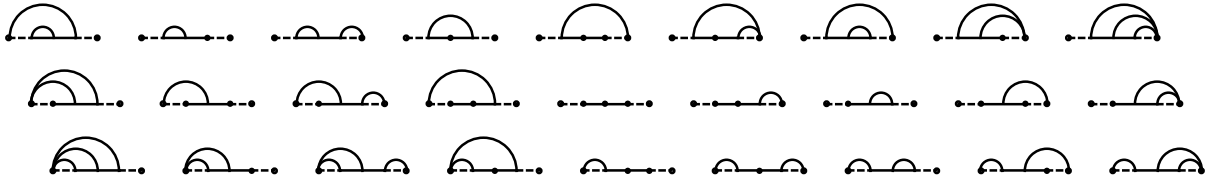


Figure 4: The basis elements of  $LP_3$ .

of the loop model corresponds to a link pattern  $\pi$ . This can be seen by erasing all closed loops in the bulk of the strip and all links connecting two vertical boundary points. The configuration on fig.(3) corresponds to the link pattern shown on fig.(5).

The object of our interest is the ground state vector  $\Psi_L$  of the transfer matrix, which we will introduce below, can be represented as a vector in the space of link patterns:

$$\Psi_L = \sum_{\pi \in LP_L} \psi_\pi |\pi\rangle. \quad (5)$$

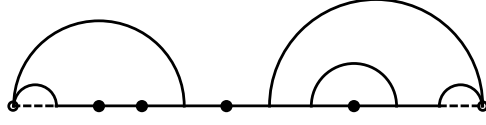


Figure 5: The link pattern that corresponds to the configuration from fig.(3).

The computation of its components  $\psi_\pi$  was the subject of our previous work.

The action of the bulk operators  $\rho^{(i)}$  on the link patterns goes as follows. An operator  $\rho^{(i)}$  non trivially acts on two neighboring vertices  $j$  and  $j + 1$  of a link pattern if the occupancy of the vertices  $j$  and  $j + 1$  coincides with the occupancy of the north west (NW) edge and the north east (NE) edge of  $\rho^{(i)}$ , respectively. Then we need to connect the middle of the NW edge of  $\rho^{(i)}$  with the  $j$ -th vertex of the link pattern and the middle of the NE edge of  $\rho^{(i)}$  with the  $j + 1$  vertex of the link pattern. In the resulting link pattern the connectivity at the points  $j$  and  $j + 1$  will be that of the middle points of the south west edge and south east edge of the  $\rho^{(i)}$  operator. Few examples are presented on fig.(6) and fig.(7). The boundary plaquettes act on the first and the last bulk points of link patterns in a similar way. Few examples of this action are presented on fig.(6). Now we need to define the  $\check{R}$ -matrix,

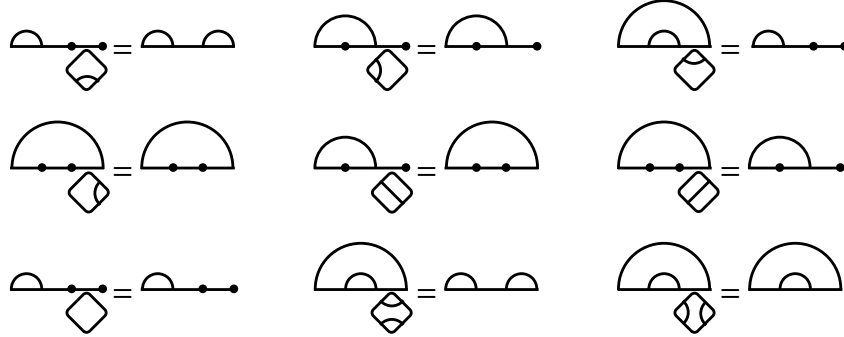


Figure 6: Few examples of the action of  $\rho^{(i)}$  operators.

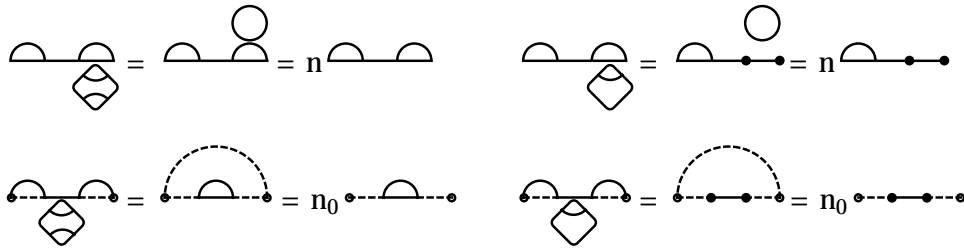


Figure 7: The operators  $\rho^{(3)}$  and  $\rho^{(8)}$  produce a closed loop or a line connecting two vertical boundaries, which is represented by the dashed semi-circle. Both, the loop weight  $n$  and the weight of the boundary to boundary line  $n_0$ , we set to 1.

$R$ -matrix, the  $K$ -matrices and then the transfer matrix of the dTL model. The  $\check{R}$ -matrix is

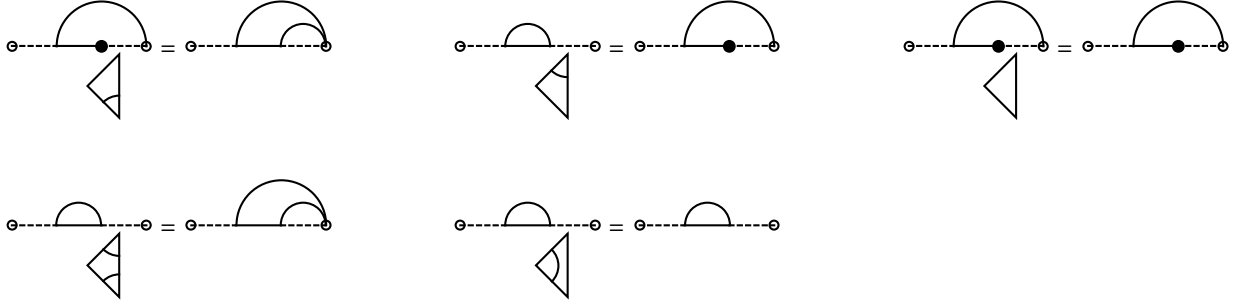


Figure 8: The action of the  $\kappa_r$ -operators.

the weighted action of  $\rho^{(i)}$ 's:

$$\check{R}_j(z_j, z_{j+1}) = \sum_{i=1}^9 \rho_j^{(i)} r_i(z_j, z_{j+1}). \quad (6)$$

Since the  $\check{R}$ -matrix acts on two points of the vector space of link patterns, it carries two

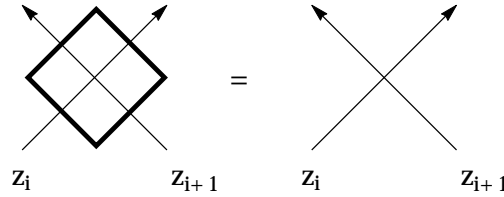


Figure 9: The  $\check{R}_i(z_i, z_{i+1})$ -matrix.

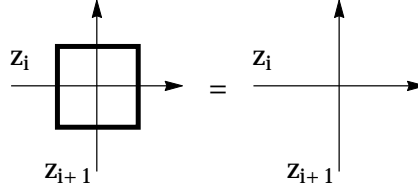


Figure 10: The  $R_i(z_i, z_{i+1})$ -matrix.

rapidity parameters  $z_j$  and  $z_{j+1}$ . On fig.(9) we show the graphical representation of the  $\check{R}$ -matrix, where the spectral parameters are carried by the straight oriented lines. To obtain the  $R$ -matrix we simply take the  $\check{R}$ -matrix in which we rotate by 45 degrees clockwise or counterclockwise the operators  $\rho^{(i)}$ , see fig.(10). The integrable  $\check{R}$ -matrix (as well as  $R$ ) depends on the ratio of two rapidities, so  $\check{R}_i(z_i, z_{i+1}) \propto \check{R}_i(z_{i+1}/z_i)$ . The integrability requires that it satisfies the Yang-Baxter (YB) equation<sup>2</sup>:

$$\check{R}_{i+1}(z/y) \check{R}_i(z/x) \check{R}_{i+1}(y/x) = \check{R}_i(y/x) \check{R}_{i+1}(z/x) \check{R}_i(z/y). \quad (7)$$

Graphically it is shown on fig.(11). This equation defines the integrable weights of the

<sup>2</sup>We choose not to spend time on the basic notions of the Yang-Baxter integrability, but instead refer to the classical literature [2].

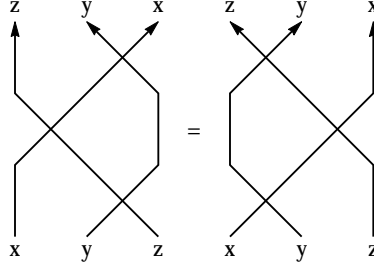


Figure 11: The Yang-Baxter equation.

$\check{R}$ -matrix:

$$\begin{aligned} r_1(z) = r_2(z) = r_3(z) = r_4(z) = \omega(\omega + 1)z, \quad r_5(z) = r_6(z) = r_7(z) = z^2 - 1, \\ r_8(z) = -(\omega + z)(\omega^2 z + 1), \quad r_9(z) = (\omega^2 + z)(\omega z + 1). \end{aligned} \quad (8)$$

Here  $\omega$  is a third root of  $-1$ ,  $\omega^3 = -1$ , which reflects the condition on the loop weight  $n = 1$ . The dTL loop model with generic value of  $n$  was obtained in [21, 20].

The  $K$ -matrix is the combination of the five boundary plaquettes. There is the left  $K$ -matrix  $K_l$  and the right  $K$ -matrix  $K_r$ :

$$K_l(z_1, x_l) = \sum_{i=1}^5 \kappa_l^{(i)} k_{i,l}(z_1, x_l), \quad K_r(z_L, x_r) = \sum_{i=1}^5 \kappa_r^{(i)} k_{i,r}(z_L, x_r) \quad (9)$$

Here,  $x_l$  and  $x_r$  play the role of the boundary rapidities. The  $K$ -matrices also have a convenient graphical representation, as shown on fig.(12). The  $R$  and  $K$ -matrices should

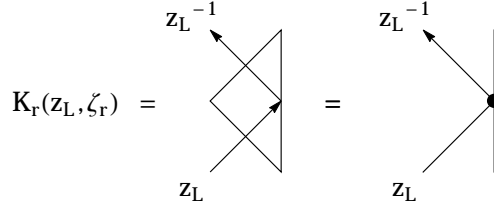


Figure 12: The operator  $K_r(z_L, x_r)$ .

satisfy the Sklyanin's reflection equation [24] also called the boundary Yang-Baxter equation (BYB). For the right boundary it reads:

$$\check{R}_{L-1}(w/z)K_r(z, x_r)\check{R}_{L-1}(1/(wz))K_r(w, x_r) = K_r(w, x_r)\check{R}_{L-1}(1/(wz))K_r(z, x_r)\check{R}_{L-1}(w/z). \quad (10)$$

and graphically is presented on fig.(13). The graphical representation of the left  $K$ -matrix as well as the corresponding reflection equation are similar to the ones of the right  $K$ -matrix. Solving the left boundary reflection equation one obtains:

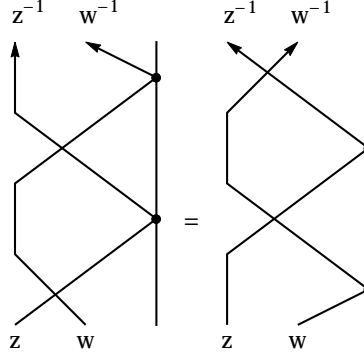


Figure 13: The boundary Yang-Baxter equation.

$$\begin{aligned}
k_{1,l}(z, x_l) &= k_{2,l}(z, x_l) = -\frac{(\omega + 1)(x_l^2 + 1)(z^2 - 1)}{xz}, \\
k_{3,l}(z, x_l) &= \frac{x_l^4 z^2 - x_l^2 z^4 + 3x_l^2 z^2 - x_l^2 + z^2}{\omega x_l^2 z^2}, \\
k_{4,l}(z, x_l) &= -\frac{(\omega + 1)(x_l^2 + 1)(z^2 - 1)(\omega - z^2)}{x_l z^2}, \\
k_{5,l}(z, x_l) &= \frac{-\omega z^4 x_l^2 + \omega x_l^2 + z^4 x_l^2 + z^2 x_l^4 + z^2}{\omega z^2 x_l^2}.
\end{aligned} \tag{11}$$

The weights of the right boundary  $K$ -matrix are given by  $k_{i,r}(z, x) = k_{i,l}(1/z, x)$ , which can be achieved by solving the right boundary reflection equation. Following the general prescription [24] we construct the double row transfer matrix (fig.(14)) using the  $R$  and  $K$ -matrices:

$$T(t|z_1, \dots, z_L; x_l, x_r) = \text{Tr}(R_1(t, z_1) \dots R_L(t, z_L) K_r(t, x_r) R_L(z_L, t^{-1}) \dots R_L(z_1, t^{-1}) K_l(t^{-1}, x_l)), \tag{12}$$

where the trace means that the lower edge of the  $K_l(t^{-1}, x_l)$  needs to be identified with the left edge of  $R_1(t, z_1)$ . The  $T$ -matrix above is the inhomogeneous transfer matrix, it depends on the bulk spectral parameters  $z_1, \dots, z_L$  associated to each space of the lattice and also on the two boundary parameters  $x_l$  and  $x_r$  associated to the left and the right boundaries. Due to the YB and the BYB two transfer matrices with different values of  $t$  commute:

$$[T(t_1), T(t_2)] = 0. \tag{13}$$

Therefore the eigenvectors of this transfer matrix must depend on  $\{z_1, \dots, z_L, x_l, x_r\}$ , but not on the parameter  $t$ . We also have the following commutation of the  $T$ -matrix with the  $\check{R}$ -matrix and the  $K$ -matrices:

$$T(t|z_1, \dots, z_i, z_{i+1}, \dots, z_L; x_l, x_r) \check{R}_i(z_i, z_{i+1}) = \check{R}_i(z_i, z_{i+1}) T(t|z_1, \dots, z_{i+1}, z_i, \dots, z_L; x_l, x_r), \tag{14}$$

$$K_l(1/z_1, x_l) T(t|1/z_1, z_2, \dots, z_L; x_l, x_r) = T(t|z_1, z_2, \dots, z_L; x_l, x_r) K_l(1/z_1, x_l), \tag{15}$$

$$K_r(1/z_L, x_r) T(t|z_1, \dots, z_{L-1}, 1/z_L; x_l, x_r) = T(t|z_1, \dots, z_{L-1}, z_L; x_l, x_r) K_r(1/z_L, x_r) \tag{16}$$

In the previous work we were focused on finding the highest eigenvector  $\Psi_L$  of the transfer matrix  $T_L$ . If we normalize properly the  $T$ -matrix we can write  $T_L \Psi_L = \Psi_L$ . Using now the



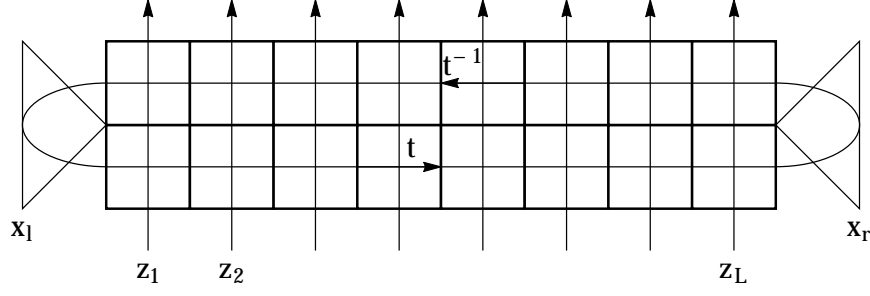


Figure 14: The graphical representation of the transfer operator.

equations (14), (15) and (16) we arrive to the qKZ equations:

$$\check{R}(z_i, z_{i+1})\Psi_L(z_1, \dots, z_i, z_{i+1}, \dots, z_L; x_l, x_r) = W(z_i, z_{i+1})\Psi_L(z_1, \dots, z_{i+1}, z_i, \dots, z_L; x_l, x_r), \quad (17)$$

$$K_l(z_1, x_l)\Psi_L(z_1, \dots, z_L; x_l, x_r) = U_l(z_1, x_l)\Psi_L(1/z_1, \dots, z_L; x_l, x_r), \quad (18)$$

$$K_r(z_L, x_r)\Psi_L(z_1, \dots, z_L; x_l, x_r) = U_r(z_L, x_r)\Psi_L(z_1, \dots, 1/z_L; x_l, x_r), \quad (19)$$

where  $W(z_i, z_{i+1})$ ,  $U_l(z_1, x_l)$  and  $U_r(z_L, x_r)$  are the normalizations of the  $\check{R}$ -matrix,  $K_l$ -matrix and  $K_r$ -matrix respectively. They can be written as combinations of weights of  $\check{R}$  and  $K_l$  and  $K_r$ , respectively, as:

$$W(z_i, z_{i+1}) = r_2(z_i, z_{i+1}) + r_6(z_i, z_{i+1}),$$

$$U_l(z_1, x_l) = k_{1,l}(z_1, x_l) + k_{3,l}(z_1, x_l) = k_{2,l}(z_1, x_l) + k_{4,l}(z_1, x_l) + k_{5,l}(z_1, x_l),$$

$$U_r(z_L, x_r) = k_{1,r}(z_L, x_r) + k_{3,r}(z_L, x_r) = k_{2,r}(z_L, x_r) + k_{4,r}(z_L, x_r) + k_{5,r}(z_L, x_r).$$

We used in our last work the equations (17)-(19) in order to compute the components  $\psi_\pi$  of the vector  $\Psi_L$ . This computation, however, is not possible without the recurrence relation which we will consider in the following section.

### 3 The first recurrence relation

In this section we will discuss the first recurrence relation (1) for the normalization  $Z_L$  of the ground state vector of the transfer matrix, which is defined as the sum of all components of  $\Psi_L$ :

$$Z_L(z_1, \dots, z_L; x_l, x_r) = \sum_{\pi \in \text{LP}_L} \psi_\pi(z_1, \dots, z_L; x_l, x_r), \quad (20)$$

The derivation of the recurrence relation of this section for  $\Psi_L$  is given in our previous work. The recurrence relation of this section follows from the factorization property of the  $R$ -matrix at a special value of its parameter. More precisely  $\check{R}(z_i\omega, z_i/\omega)$  factorizes into two operators

$$\check{R}_i(z\omega, z/\omega) = (\omega^2 + \omega)z^2 S_i M_i. \quad (21)$$

This gives rise to a "modified" version of the YB equation. It involves two  $R$ -operators and one  $M$ -operator:

$$M_i \check{R}_i(t, z_i\omega) R_{i+1}(t, z_i/\omega) = R_i(t, z_i) M_i. \quad (22)$$

In the quantum group literature this is the quasi-triangularity condition of the corresponding Hopf algebra. The operator  $M$  maps two sites into one site and hence merges the two  $R$ -matrices into one after the substitution  $z_i = z_i\omega$  and  $z_{i+1} = z_i/\omega$ . The graphical representation of  $M$ ,  $S$  and the equation (22) are presented on the figures (15) and (16). Now

$$M_i = \triangle + \triangle + \triangle + \triangle \quad S_i = \nabla + \nabla + \nabla + \nabla$$

Figure 15: The  $M_i$  and  $S_i$  operators.

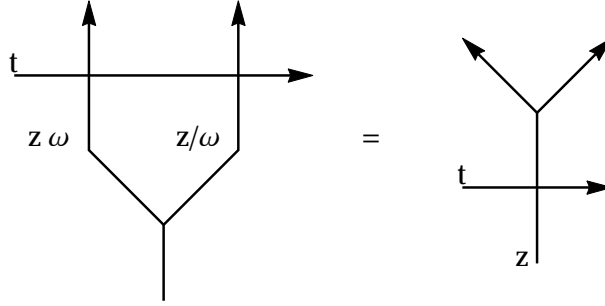


Figure 16: The graphical version of the equation (22). The  $M$ -operator is represented by the trivalent vertex.

if we apply  $M_i$  to the transfer matrix using (22) we get:

$$M_i T_{L+1}(t|z_1, \dots, z_i\omega, z_i/\omega, z_{i+1}, \dots, z_L; x_l, x_r) = T_L(t|z_1, \dots, z_i, z_{i+1}, \dots, z_L; x_l, x_r) M_i, \quad (23)$$

Applying eq.(23) to the ground state we find the desired recurrence relation:

$$M_i \Psi_{L+1}(z_1, \dots, z_i\omega, z_i/\omega, z_{i+1}, \dots, z_L; x_l, x_r) = F_i(z_1, \dots, z_L; x_l, x_r) \Psi_L(z_1, \dots, z_i, z_{i+1}, \dots, z_L; x_l, x_r). \quad (24)$$

The index  $i$  in  $F_i$  signifies that  $F$  has a special dependence on the variable  $z_i$ . The explicit form of this polynomial we found in our first paper on the dTL model, it reads:

$$F_i(z_1, \dots, z_L; x_l = z_0, x_r = z_{L+1}) = \prod_{0 \leq j \neq i \leq L+1} \frac{(z_i + z_j)(z_i z_j + 1)}{z_i z_j}. \quad (25)$$

The recurrence eq.(24) relates the components of  $\Psi_{L+1}$  to the components of  $\Psi_L$ . In particular, if  $\pi \in \text{LP}_{L+1}$  has two empty sites at the positions  $i$  and  $i+1$ , i.e.  $\pi = \{\alpha, 0, 0, \beta\}$  and  $\alpha \in \text{LP}_{i-1}$  and  $\beta \in \text{LP}_{L-i}$ , then the recurrence (24) maps  $\psi_{\{\alpha, 0, 0, \beta\}}$  to  $\psi_{\{\alpha, 0, \beta\}}$ . When all the sites are empty in  $\pi$  we have:

$$\psi_e(z_1, \dots, z_i\omega, z_i/\omega, z_{i+1}, \dots, z_L; x_l = z_0, x_r = z_{L+1}) = \prod_{0 \leq j \neq i \leq L+1} \frac{(z_i + z_j)(z_i z_j + 1)}{z_i z_j} \psi_e(z_1, \dots, z_i, \dots, z_L; z_0, z_{L+1}). \quad (26)$$

If the occupancy in  $\pi$  is fixed the sum of all  $\psi_\pi$  with such occupancy is equal to  $\psi_e$ . This is a consequence of the fact that the transfer matrix is a stochastic matrix. There are in total  $2^L$  different choices of the occupancy for the link patterns in  $\text{LP}_L$ , hence  $Z_L = 2^L \psi_e$ . We will omit this constant and simply consider the equation:

$$Z_L(z_1, \dots, z_i \omega, z_i / \omega, z_{i+1}, \dots, z_L; z_0, z_{L+1}) = \prod_{0 \leq j \neq i \leq L+1} \frac{(z_i + z_j)(z_i z_j + 1)}{z_i z_j} Z_L(z_1, \dots, z_i, \dots, z_L; z_0, z_{L+1}). \quad (27)$$

Let us now examine the symmetries of  $Z_L$ . First of all  $Z_L$  is symmetric in  $\{z_1, \dots, z_L\}$ . This can be seen using the qKZ equation (17) for the components  $\psi_{\alpha, n_i, n_{i+1}, \beta}(\dots, z_i, z_{i+1}, \dots)$  with  $\alpha \in \text{LP}_{i-1}$  and  $\beta \in \text{LP}_{L-i-1}$  (for more discussions of the qKZ equation see [9], and for the dTL [12]). Let us consider the following sum:

$$\bar{\psi}_{n_i, n_{i+1}} = \sum_{\substack{\alpha \in \text{LP}_{i-1} \\ \beta \in \text{LP}_{L-i-1}}} \psi_{\alpha, n_i, n_{i+1}, \beta}. \quad (28)$$

Assuming  $W = W(z_i, z_{i+1})$ ,  $r_j = r_j(z_i, z_{i+1})$ ,  $k_{l,j} = k_{l,j}(z_i, z_{i+1})$ ,  $\bar{\psi}_{1,-1} = \bar{\psi}_{1,-1}(z_i, z_{i+1})$  and  $\tilde{\bar{\psi}}_{1,-1} = \tilde{\bar{\psi}}_{1,-1}(z_{i+1}, z_i)$ , the qKZ equation for each combination of  $n_i$  and  $n_{i+1}$  gives:

$$\begin{aligned} W \tilde{\bar{\psi}}_{1,-1} &= r_9 \bar{\psi}_{1,-1} + r_1 \bar{\psi}_{0,0} + r_8 (\bar{\psi}_{1,1} + \bar{\psi}_{-1,1} + \bar{\psi}_{1,-1} + \bar{\psi}_{-1,-1}), \\ W \tilde{\bar{\psi}}_{1,1} &= r_9 \bar{\psi}_{1,1}, \quad W \tilde{\bar{\psi}}_{-1,-1} = r_9 \bar{\psi}_{-1,-1}, \quad W \tilde{\bar{\psi}}_{-1,1} = r_9 \bar{\psi}_{-1,1}, \\ W \tilde{\bar{\psi}}_{0,0} &= r_7 \bar{\psi}_{0,0} + r_3 (\bar{\psi}_{1,1} + \bar{\psi}_{-1,1} + \bar{\psi}_{1,-1} + \bar{\psi}_{-1,-1}) \end{aligned}$$

Since  $W = r_3 + r_8 + r_9 = r_2 + r_7$  the sum of these five equations gives:

$$\tilde{\bar{\psi}}_{1,1} + \tilde{\bar{\psi}}_{1,-1} + \tilde{\bar{\psi}}_{-1,1} + \tilde{\bar{\psi}}_{-1,-1} + \tilde{\bar{\psi}}_{0,0} = \bar{\psi}_{1,1} + \bar{\psi}_{-1,1} + \bar{\psi}_{1,-1} + \bar{\psi}_{-1,-1} + \bar{\psi}_{0,0}. \quad (29)$$

If we take now the remaining equations

$$\begin{aligned} W \tilde{\bar{\psi}}_{0,1} &= r_4 \bar{\psi}_{0,1} + r_6 \bar{\psi}_{1,0}, \\ W \tilde{\bar{\psi}}_{1,0} &= r_2 \bar{\psi}_{1,0} + r_5 \bar{\psi}_{0,1}, \\ W \tilde{\bar{\psi}}_{0,-1} &= r_4 \bar{\psi}_{0,-1} + r_6 \bar{\psi}_{-1,0}, \\ W \tilde{\bar{\psi}}_{-1,0} &= r_2 \bar{\psi}_{-1,0} + r_5 \bar{\psi}_{0,-1}, \end{aligned}$$

we obtain a similar result:

$$\tilde{\bar{\psi}}_{0,1} + \tilde{\bar{\psi}}_{1,0} = \bar{\psi}_{0,1} + \bar{\psi}_{1,0}, \quad \tilde{\bar{\psi}}_{0,-1} + \tilde{\bar{\psi}}_{-1,0} = \bar{\psi}_{0,-1} + \bar{\psi}_{-1,0}, \quad (30)$$

due to the fact:  $W = r_2 + r_6 = r_4 + r_5$ . Summing up the equations (29) and (30) gives us the desired symmetry of  $Z_L$  in the interchange of  $z_i$  and  $z_{i+1}$ .

Similarly we prove that  $Z_L(1/z_1, \dots) = Z_L(z_1, \dots)$  using the left boundary qKZ equation (18). Summing the three equations for  $n_1 = 1, 0, -1$  for  $\bar{\psi}_{n_1} = \sum_{\alpha} \psi_{n_1, \alpha}$  gives:

$$U_l(\tilde{\bar{\psi}}_1 + \tilde{\bar{\psi}}_0 + \tilde{\bar{\psi}}_{-1}) = (\kappa_l^{(2)} + \kappa_l^{(4)} + \kappa_l^{(5)})(\bar{\psi}_1 + \bar{\psi}_{-1}) + (\kappa_l^{(1)} + \kappa_l^{(3)})\bar{\psi}_0, \quad (31)$$

where now  $\tilde{\psi}_1 = \bar{\psi}_1(1/z_1, \dots)$ . Noticing again that  $U_l = k_1 + k_3 = k_2 + k_4 + k_5$  finishes the argument.

In the course of the computation of the components of  $\Psi_L$  we observed that the boundary spectral parameters appear in the components  $\psi_\pi$  in a similar way as the bulk spectral parameters. In particular, the  $\psi_e$  element is symmetric in the full set of parameters  $\{x_l^{\pm 1}, z_1^{\pm 1}, \dots, z_L^{\pm 1}, x_r^{\pm 1}\}$  and the recurrence (1) can be also applied to  $x_l$  and  $x_r$ . The proof of this will appear in [6].

Finally, the recurrence relation (1) has the initial condition  $Z_0 = 1$ . Considering  $Z_L$  as a polynomial  $Z_L(w)$ , say, with  $z_1 = w$ , with the degree growth  $2L$ , we find that the recurrence (1) fixes the values of the polynomial  $Z_L(w)$  at  $2L + 2$  points, i.e. when  $w = z_i/\omega^2$  and  $w = \omega^2/z_i$  for all  $i > 1$  and also for  $x_l = z_0$  and  $x_r = z_{L+1}$ . This fixes  $Z_L$  uniquely by the polynomial interpolation formula.

Now let us briefly mention the recurrence relation and its solution for the periodic model. It was found in [7] that the sum of the ground state components of the dTL model on a cylinder of circumference  $L$  satisfies:

$$Z_L^p(z_1, \dots, z_{L-1} = z_{L-1}\omega, z_L = z_{L-1}/\omega) = F_{L-1}^p(z_1, \dots, z_{L-1}) Z_{L-1}^p(z_1, \dots, z_{L-1}), \quad (32)$$

with  $F^p$ :

$$F_i^p(z_1, \dots, z_L) = z_i \prod_{1 \leq j \neq i \leq L} (z_i + z_j). \quad (33)$$

$Z_L^p$  is symmetric under the interchange of the rapidities but not under their inversion. It has a similar initial condition  $Z_1^p = 1$ , and is the unique solution of (32). The form (33) of the recurrence factor  $F^p$  suggests that a good basis to express the solution of  $Z_L^p$  is the set of elementary symmetric polynomials for which  $F^p$  is the generating function. These polynomials are defined as follows

$$\begin{aligned} E_m(z_1, \dots, z_L) &= \sum_{1 \leq i_1 < \dots < i_m \leq L} z_{i_1} \dots z_{i_m}, \\ E_m(z_1, \dots, z_L) &= 0 \quad \text{for } m < 0, \quad \text{and } m > L. \end{aligned} \quad (34)$$

The determinant:

$$Z_L^p = \det_{1 \leq i, j \leq L-1} E_{3j-2i}(z_1, \dots, z_L) \quad (35)$$

solves the recurrence (32) [7]. This can also be written as:

$$Z_L^p = E_1(z_1, \dots, z_L) \det_{1 \leq i, j \leq L-2} E_{3j-2i}(z_1, \dots, z_L) \quad (36)$$

In fact, for further convenience we prefer to ignore the prefactor  $E_1$  and by abuse of notation we will still call the resulting polynomial  $Z_L^p$ . Recall the Jacobi-Trudi identity for the skew Schur polynomial  $S_{\lambda/\mu}$  [14], where  $\lambda$  and  $\mu$  are two partitions such that each part  $\mu_i$  of  $\mu$  is no greater than the same part  $\lambda_i$  of  $\lambda$ , which has the length denoted by  $|\lambda|$ :

$$S_{\lambda/\mu} = \det_{1 \leq i, j \leq |\lambda|} E_{\lambda'_i - \mu'_j - i + j}. \quad (37)$$

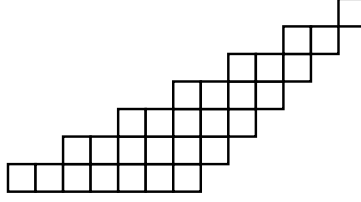


Figure 17: The skew diagram for the partition  $\lambda/\mu$ , where  $\lambda = \{13, 12, 11, 10, 9, 8, 7\}$  and  $\mu = \{12, 10, 8, 6, 4, 2, 0\}$ .

The primed partitions are the transposed partitions of the unprimed. If we choose  $\lambda'_i = 2L - i$  and  $\mu'_j = 2L - 2j$  then we get (35). Such a skew partition corresponds to the Young diagram that looks like a staircase, see fig.(17). One can as well write the formula (35) for  $Z_L^p$  in terms of the homogeneous symmetric functions, however, we prefer to stick in what follows to the elementary symmetric polynomials.

Let us get back to (27). By the analogy with the dense TL model [5, 3] the solution to this recurrence relation should involve the symplectic version of  $S_{\lambda/\mu}$ . However, to the authors knowledge the discussion of the skew symplectic Schur functions is missing in the literature. The form of the  $F_i$  function suggests, in turn, that a good basis for  $Z_L$  is the set of the elementary symmetric polynomials with extended list of arguments, which includes the symmetry in the inversion of the rapidities, i.e.:

$$\varepsilon_m(z_1, \dots, z_L) = E_m(z_1, \dots, z_L, 1/z_1, \dots, 1/z_L). \quad (38)$$

These symmetric polynomials are related to the elementary symmetric polynomials of  $z_i$ 's and their inverses separately through the formula:

$$\varepsilon_m(z_1, \dots, z_L) = \sum_{n=0}^L E_{L-n}(z_1, \dots, z_L) E_{L+n-m}(1/z_1, \dots, 1/z_L). \quad (39)$$

There exists a determinant of a matrix of  $\varepsilon_m$ 's which is the solution of (27), however, the corresponding matrix is very complicated. We will comment on that in the end of this section.  $Z_L$  can be alternatively written as a determinant of a simple matrix of  $\varepsilon_m$ 's divided by a certain symmetric polynomial. Let us show how to get this expression. First, let us consider

$$\tilde{Z}_L^p(z_1, \dots, z_L) = Z_{2L}^p(z_1, \dots, z_L, 1/z_1, \dots, 1/z_L) = \det_{1 \leq i, j \leq 2L-2} \varepsilon_{3j-2i}(z_1, \dots, z_L). \quad (40)$$

It satisfies the following recurrence relation:

$$\begin{aligned} \tilde{Z}_L^p(z_1, \dots, z_{L-1} = z\omega, z_L = z/\omega) &= (z + \frac{1}{z})(z^2 + 1 + \frac{1}{z^2}) \\ &\prod_{1 \leq j \neq i \leq L-2} \frac{(z + z_j)^2 (zz_j + 1)^2}{z^2 z_j^2} \tilde{Z}_{L-1}^p(z_1, \dots, z_{L-2}, z). \end{aligned} \quad (41)$$

A  $(2L - 2) \times (2L - 2)$  matrix  $\varepsilon_{3j-2i}$  is centrosymmetric. Indeed, a centrosymmetric matrix of size  $L \times L$  by definition is a matrix with the following symmetry

$$M_{j,i} = M_{L-j+1, L-i+1}, \quad (42)$$

For example, when  $L = 4$   $M$  has the following entries:

$$\begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{2,4} & m_{2,3} & m_{2,2} & m_{2,1} \\ m_{1,4} & m_{1,3} & m_{1,2} & m_{1,1} \end{pmatrix}$$

In our case it means:

$$\varepsilon_{3j-2i} = \varepsilon_{3((2L-2)-j+1)-2((2L-2)-i+1)} = \varepsilon_{2L-2-(3j-2i)+1}. \quad (43)$$

The polynomials  $\varepsilon_m$  are generated by  $F$ :

$$F_{L+1}(z_1, \dots, z_L, z_{L+1} = t) = \sum_{i=0}^{2L} t^{i-L} \varepsilon_i(z_1, \dots, z_L). \quad (44)$$

which implies the symmetry in (43). As the matrix  $\varepsilon_{3j-2i}$  is centrosymmetric it is then block diagonalized by the following transformation:

$$T = \begin{pmatrix} -I & J \\ I & J \end{pmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{2} \begin{pmatrix} -I & I \\ J & J \end{pmatrix}, \quad (45)$$

where  $I$  is the  $L-1 \times L-1$  unit matrix and  $J$  is the  $L-1 \times L-1$  matrix with elements equal to 1 on the counterdiagonal and all other elements are zero. For example, applying this transformation to a centrosymmetric matrix of size 4 gives:

$$TMT^{-1} = \begin{pmatrix} m_{1,1} - m_{1,4} & m_{1,2} - m_{1,3} & 0 & 0 \\ m_{2,1} - m_{2,4} & m_{2,2} - m_{2,3} & 0 & 0 \\ 0 & 0 & m_{1,1} + m_{1,4} & m_{1,2} + m_{1,3} \\ 0 & 0 & m_{2,1} + m_{2,4} & m_{2,2} + m_{2,3} \end{pmatrix}$$

Now applying this transformation to the matrix  $\varepsilon_{3j-2i}$  under the sign of the determinant we get

$$\det_{1 \leq k, l \leq 2L-2} T_{k,i} \varepsilon_{3j-2i} T_{j,l}^{-1} = \frac{1}{2} \det_{1 \leq k, l \leq L-1} (\varepsilon_{3j-2i} - \varepsilon_{3j+2i-4L}) \det_{L \leq i, j \leq 2L-2} (\varepsilon_{3j-2i} + \varepsilon_{3j+2i-4L}). \quad (46)$$

These determinants define two symmetric polynomials:

$$V_L(z_1, \dots, z_L) = \det_{1 \leq i, j \leq L-1} (\varepsilon_{3j-2i} - \varepsilon_{3j+2i-4L}), \quad (47)$$

$$W_L(z_1, \dots, z_L) = \frac{1}{2} \det_{L \leq i, j \leq 2L-2} (\varepsilon_{3j-2i} + \varepsilon_{3j+2i-4L}). \quad (48)$$

By the degree growth argument they satisfy the recurrence relations:

$$V_L(z\omega, z/\omega, z_3, \dots, z_L) = \prod_{3 \leq i \leq L} \frac{(z_i + z)(zz_i + 1)}{z_i z} \left(z + \frac{1}{z}\right) V_{L-1}(z, z_3, \dots, z_L), \quad (49)$$

$$W_L(z\omega, z/\omega, z_3, \dots, z_L) = \prod_{3 \leq i \leq L} \frac{(z_i + z)(zz_i + 1)}{z_i z} \left(z^2 + 1 + \frac{1}{z^2}\right) W_{L-1}(z, z_3, \dots, z_L). \quad (50)$$

Alternatively, one could prove that (47) satisfies (49) and (48) satisfies (50) using the appropriate row-column manipulation in the corresponding matrices. We will use this method in the next chapter to prove a similar statement for other determinants and recurrence relations. Let us look at the eq.(49). Once the initial conditions are set, there is a unique polynomial that solves this recurrence relation. If the initial condition is  $V_2(x, y) = \varepsilon_1(x, y)$ , then the solution is precisely the polynomial  $V_L$  defined by (47). On the other hand, if we take  $Z_L$  and multiply it by a polynomial  $P_L^p$  that satisfies:

$$P_L^p(z\omega, z/\omega, z_3, \dots, z_L) = (z + \frac{1}{z})P_{L-1}^p(z, z_3, \dots, z_L). \quad (51)$$

and has the appropriate initial condition, i.e.  $P_2^p(x, y) = \varepsilon_1(x, y)$  which coincides with  $V_2$ , then the product  $Z_L(z_1, \dots, z_L)P_L^p(z_1, \dots, z_L)$  also satisfies the recurrence relation (49). By the uniqueness of the solution of the recurrence relation (49) this means  $P_L^p$  divides  $V_L$  and we obtain the sum rule  $Z_L$ :

$$Z_L(z_1, \dots, z_L) = \frac{\det_{1 \leq i, j \leq L-1} (\varepsilon_{3j-2i} - \varepsilon_{3j+2i-4L})}{P_L^p(z_1, \dots, z_L)}, \quad (52)$$

where  $P_L^p$  can be written compactly as:

$$P_L^p(z_1, \dots, z_L) = \frac{i^{L+1}}{2(\omega - \omega^{-1})} \left\{ \prod_{j=1}^L \frac{(\omega z_j + i)(\omega^{-1} z_j - i)}{z_j} - \prod_{j=1}^L \frac{(\omega^{-1} z_j + i)(\omega z_j - i)}{z_j} \right\}, \quad (53)$$

where  $i$  is the imaginary unit  $i^2 = -1$ . One can now easily check the equation (51). We can alternatively look for a polynomial that satisfies

$$P_L(z\omega, z/\omega, z_3, \dots, z_L) = (z^2 + 1 + \frac{1}{z^2})P_{L-1}(z, z_3, \dots, z_L), \quad (54)$$

and has the initial condition that of  $W_L$  in (48). Such polynomial exists:

$$P_L(z_1, \dots, z_L) = \frac{(-1)^L \omega}{2(1 - \omega^2)(\omega - \omega^{-1})} \left\{ \prod_{j=1}^L \frac{(\omega + \omega z_j)(\omega + \omega z_j)}{z_j \omega} \frac{(\omega + \omega^2 z_j)(\omega^2 + \omega z_j)}{z_j \omega} \right. \\ \left. - \prod_{j=1}^L \frac{(\omega + \omega^{-1} z_j)(\omega^{-1} + \omega z_j)}{z_j \omega} \frac{(\omega + \omega^{-2} z_j)(\omega^{-2} + \omega z_j)}{z_j \omega} \right\}. \quad (55)$$

The product  $W_L P_L$  satisfies the recurrence (50), hence:

$$Z_L(z_1, \dots, z_L) = \frac{\det_{1 \leq i, j \leq L-1} (\varepsilon_{3j-2i} + \varepsilon_{3j+2i-4L})}{P_L(z_1, \dots, z_L)}. \quad (56)$$

Last remark of this section follows. The symmetric polynomial  $P_L^p$  can be written as a ratio of two determinants:  $V_{L+2}(z_1, \dots, z_L, \omega, \omega^2)/W_L(z_1, \dots, z_L)$ , and since both  $V_L$  and  $W_L$  are determinants the sum rule  $Z_L$  is also a determinant of a matrix of the symmetric polynomials  $\varepsilon_m$ . This determinant is hard to write in a closed form, however. In the next section we will use a different recurrence relation and hence different basis of symmetric polynomials to express both  $Z_L$  and  $Z_L^p$  in determinantal forms. Surprisingly, the second type of recurrence relations for periodic and open  $Z_L$  are defined via a slightly more general versions of the polynomials  $P_L^p$  and  $P_L$ . We do not give the proofs of the recurrence relations in the next section. However, we believe they can be obtained, in particular, by studying the IK vertex model. This question will be addressed in [11].

## 4 The second recurrence relation

Let us start with the sum rule  $Z_L^p$  of the periodic dTL  $O(1)$  ground state. Another recurrence relation for  $Z_L^p$  appears in [7] without a proof. This recurrence relation is unrelated to the one we discussed previously. To write it we first extend the definition of  $P_L^p$  symmetric polynomial in (53) to:

$$P_L^p(z_1, \dots, z_L | t) = \frac{t}{2(\omega - \omega^{-1})} \left\{ \prod_{j=1}^L (\omega z_j + t)(\omega^{-1} z_j - t) - \prod_{j=1}^L (\omega^{-1} z_j + t)(\omega z_j - t) \right\}, \quad (57)$$

where we omitted the  $E_L$  in the denominator in eq.(53) and included another variable  $t$ . The recurrence relation reads:

$$Z_L^p(z_1, \dots, z_{L-1} = t, z_L = -t) = P_{L-2}^p(z_1, \dots, z_{L-2} | t) Z_{N-2}^p(z_1, \dots, z_{L-2}). \quad (58)$$

Once again we suspect that a good basis of symmetric polynomials to express the solution is to use  $P^p$  as a generating function. Looking at the coefficients of  $t$  in  $P^p$

$$P_L^p(z_1, \dots, z_L | t) = \frac{1}{2(\omega - \omega^{-1})} \sum_{n_1, n_2=0}^L (-1)^{n_1} t^{n_1+n_2+1} E_{L-n_1} E_{L-n_2} (\omega^{n_1-n_2} - \omega^{-n_1+n_2}), \quad (59)$$

we notice that, because of the symmetry in the interchange of  $n_1$  and  $n_2$  in this sum, only even powers of  $t$  enter this expansion, hence it can be written as:

$$P_L^p(z_1, \dots, z_L | t) = \sum_{i=1}^L t^{2i} \mu_{L-i+1}(z_1, \dots, z_L). \quad (60)$$

which defines the symmetric polynomials quadratic in  $E_m$ 's:

$$\mu_i = \frac{1}{2(\omega - \omega^{-1})} \sum_{m=0}^L (-1)^{L+n} (\omega^{2(i-m)-1} - \omega^{2(m-i)+1}) E_m E_{2i-m-1}, \quad (61)$$

for  $i = 1, \dots, L$  and otherwise 0. We found that the solution to the equation (58) is the following determinant written in the basis of symmetric polynomials  $\mu_i$ <sup>3</sup>:

$$Z_L^p(z_1, \dots, z_L) = \det_{0 \leq i, j \leq L/2-1} \mu_{3i-j+1}(z_1, \dots, z_L), \quad \text{for even } L, \quad (62)$$

$$Z_L^p(z_1, \dots, z_L) = \det_{1 \leq i, j \leq (L-1)/2} \mu_{3i-j}(z_1, \dots, z_L), \quad \text{for odd } L. \quad (63)$$

Before proving this we need to examine the properties of the symmetric polynomials  $\mu_i$ . In particular, how do they behave under the relevant recursion. This is best seen by looking at the behavior of  $P^p$  under the substitution  $z_L = z$  and  $z_{L-1} = -z$ :

$$P_L^p(z_1, \dots, z_{L-1} = -z, z_L = z | t) = (t^4 + t^2 z^2 + z^4) P_{L-2}^p(z_1, \dots, z_{L-2} | t). \quad (64)$$

---

<sup>3</sup>This is not a basis of the space of symmetric polynomials, it is rather a nonlinear basis in which  $Z_L^p$  can be expressed.



Comparing this to (60) allows to write the recurrence relation satisfied by the  $\mu_i$ 's:

$$\mu_i(z_1, \dots, z_{L-1} = -z, z_L = z) = z^4 \mu_{i-2}(z_1, \dots, z_{L-2}) + z^2 \mu_{i-1}(z_1, \dots, z_{L-2}) + \mu_i(z_1, \dots, z_{L-2}). \quad (65)$$

Now we can use row column manipulations to prove, for example, that (62) satisfies (58). The proof for the odd  $L$  goes in a similar manner. First, we apply the substitution (65) in the matrix  $\mu_{3i-j+1}$ , which brings it to the form

$$\tilde{\mu}_{3i-j+1} = z^4 \mu_{3i-j-1}(z_1, \dots, z_{L-2}) + z^2 \mu_{3i-j}(z_1, \dots, z_{L-2}) + \mu_{3i-j+1}(z_1, \dots, z_{L-2}). \quad (66)$$

Then we subtract each column  $j+1$  multiplied by  $z^2$  from the column  $j$  starting with  $j=1$  up to  $j=L/2-2$ . After this manipulation the matrix elements become:

$$-z^6 \mu_{3i-j-2}(z_1, \dots, z_{L-2}) + \mu_{3i-j+1}(z_1, \dots, z_{L-2}). \quad (67)$$

Finally, we add each row  $i$  multiplied by  $z^6$  to the row  $i+1$ , starting with  $i=1$  up to  $i=L/2-2$ . Note, in the first row there is only one nonzero element, i.e.  $\mu_1$ , which was unaffected by the substitution (65), neither by the first column manipulation since the other elements in this row are zero. Therefore, after the first row manipulation the second row is left with only one term:  $\mu_4$ . Adding this multiplied by  $z^6$  to the third row leaves it with  $\mu_7$ , and so on. Similar subtraction happens in the other columns. We are left then with the desired matrix  $\mu_{3i-j+1}$  occupying first  $L/2-2$  rows and  $L/2-2$  columns. The elements of the last row are equal to zero except for the one that is in the last column, i.e. at the position  $L/2-1, L/2-1$ . For the convenience we will use an integer  $m$  instead of  $L/2$ . This matrix element is equal to the  $P^p$  polynomial in the form (60). The row manipulation essentially means that we multiply each element of a row  $j$  by  $z^{6(m-1-j)}$  and then add them up starting from the above element, the last element in the column is then the sum of all the rest elements in this column thus multiplied, so we have

$$\begin{aligned} \sum_{i=i_0}^{m-1} z^{6(m-1-i)} (\mu_{3i-m+2} + z^2 \mu_{3i-m+1} + z^4 \mu_{3i-m}) &= z^2 \mu_{2m-2}(z_1, \dots, z_{2m-2}) + \\ &+ z^4 \mu_{2m-3}(z_1, \dots, z_{2m-2}) + z^6 (\mu_{2m-4}(z_1, \dots, z_{2m-2}) + z^2 \mu_{2m-5}(z_1, \dots, z_{2m-2}) + \\ &+ z^4 \mu_{2m-6}(z_1, \dots, z_{2m-2})) + \dots = P_p(z_1, \dots, z_{2m-2}|z), \end{aligned} \quad (68)$$

where  $i_0$  is the position of the first non vanishing entry from above in the last column. This completes the proof.

One can alternatively view this row column manipulation as acting on the matrix with the entries (66) from the left by the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z^6 & 1 & 0 & \dots & 0 \\ z^{12} & z^6 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ z^{6(m-1)} & z^{6(m-2)} & z^{6(m-3)} & \dots & 1 \end{pmatrix}, \quad (69)$$

and also acting from the right by the matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -z^2 & 1 & 0 & \dots & 0 \\ 0 & -z^2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (70)$$

we have:

$$\det_{0 \leq k, l \leq L/2-1} A_{k,i} \tilde{\mu}_{3i-j+1} B_{j,l} = P_{L-2}^p \det_{0 \leq i, j \leq L/2-2} \mu_{3i-j+1}. \quad (71)$$

We are allowed to act with these matrices on (66) under the sign of determinant since  $\det A = 1$  and  $\det B = 1$ .

It seems that the determinants in (62) and (63) must be related to (35) by some transformation, as they give the same polynomial. Unfortunately, we are not aware of this transformation at this point. Let us turn now to the discussion of the second recurrence relation applied to the  $Z_L$  sum rule for the open boundary dTL model.

The derivation of the second recurrence relation (2) for the open boundary case is essentially the same as for the periodic, hence we refer again to [11]. For now we will assume the validity of (2) and continue with solving the second recurrence relation for the sum rule  $Z_L$  of the open loop model.  $Z_L$  satisfies the recurrence (2) if the polynomial  $P_L$  is the following:

$$P_L(z_1, \dots, z_L | t) = \frac{(-1)^L t}{2(1-t^2)(\omega - \omega^{-1})} \left\{ \prod_{j=1}^L \frac{(t + \omega z_j)(\omega + t z_j)}{z_j t} \frac{(t + \omega^2 z_j)(\omega^2 + t z_j)}{z_j t} \right. \\ \left. - \prod_{j=1}^L \frac{(t + \omega^{-1} z_j)(\omega^{-1} + t z_j)}{z_j t} \frac{(t + \omega^{-2} z_j)(\omega^{-2} + t z_j)}{z_j t} \right\}. \quad (72)$$

This polynomial itself satisfies a few recurrence relations. First of all:

$$P_L(z_1, \dots, z_{L-1} = z_{L-1}\omega, z_L = z_{L-1}/\omega | t) = (z_{L-1}^2 + \frac{1}{z_{L-1}^2} - t^2 - \frac{1}{t^2}) P_{L-1}(z_1, \dots, z_{L-1} | t). \quad (73)$$

If we set in this equation  $t = \omega$ , then it reproduces eq.(54).

Once again we consider  $P_L$  as a generating function of the symmetric polynomials which form a convenient basis to solve the recurrence (2) for  $Z_L$ . First we expand it in the elementary symmetric polynomials:

$$P_L(z_1, \dots, z_L | t) = \frac{(-1)^L t^{-2L}}{2(\omega - \omega^{-1})(t^2 - 1)} \sum_{s=0}^{2L} t^{2s} \sum_{r=0}^{2L} (-1)^r (\omega^{2(r-s)+1} - \omega^{-2(r-s)-1}) \times \\ \sum_{m,l=0}^L E_{L-l}(1/z_1, \dots, 1/z_L) E_{L+l-(2s-r-1)}(z_1, \dots, z_L) E_{L-m}(z_1, \dots, z_L) E_{L+m-r}(1/z_1, \dots, 1/z_L), \quad (74)$$

which can be rewritten in terms of  $\varepsilon_i$  using (39):

$$P_L(z_1, \dots, z_L | t) = \frac{(-1)^L t^{-2L}}{2(\omega - \omega^{-1})(t^2 - 1)} \sum_{s=0}^{2L} t^{2s} \sum_{r=0}^{2L} (-1)^r (\omega^{2(r-s)+1} - \omega^{-2(r-s)-1}) \varepsilon_r \varepsilon_{2s-1-r}. \quad (75)$$

We define a new set of symmetric polynomials  $\nu_i$ :

$$\nu_i = \sum_{j=0}^{2L} (-1)^{j+i} \varepsilon_{2i-1-j} \varepsilon_j \frac{(\omega^{2(j-i)+1} - \omega^{-2(j-i)-1})}{2(\omega - \omega^{-1})}. \quad (76)$$

These symmetric polynomials are defined by the same formula as  $\mu_i$  (61) up to an overall minus sign and the replacement:  $E \rightarrow \varepsilon$ .

One way to find a determinantal expression for  $Z_L$  is to use the same transformation as before (45) applied to (62) where  $E$  is replaced by  $\varepsilon$ . Although the matrix  $\mu_{3i-j+1}$  is not centrosymmetric one can, however, make it almost centrosymmetric by interchanging some columns. After such manipulations and a little bit of algebra one obtains:

$$Z_L(z_1, \dots, z_L) = \det_{0 \leq i, j \leq m-1} (\nu_{3i-j+1} - \nu_{3i+j+3-L}), \quad \text{for } L = 2m + 1, \quad (77)$$

$$Z_L(z_1, \dots, z_L) = \frac{1}{P_L(z_1, \dots, z_L | \omega)} \det_{0 \leq i, j \leq m-1} (\nu_{3i-j+2} - \nu_{3i+j+2-L}), \quad \text{for } L = 2m, \quad (78)$$

which can be proven by row column manipulations similarly as before. It is not evident how to write the formula for even  $L$  in a pure determinant form using the  $\nu_i$  functions. One could try to improve it, however, we can do better if we use different symmetric functions instead of  $\nu_i$ . Indeed, we can simply follow the idea that we used for the periodic cases, i.e. to use symmetric functions generated by  $P_L$ :

$$P_L(z_1, \dots, z_L | t) = \sum_{i=1}^{L-1} (t^{2i} + t^{-2i}) \lambda_i + \lambda_0, \quad (79)$$

where  $\lambda_i$  can be written in terms of  $\nu_i$ 's as follows:

$$\lambda_i = \sum_{k=i}^{L-1} (-1)^k \nu_{L-k}. \quad (80)$$

The set of polynomials  $\lambda$  is a more natural basis for  $Z_L$  than the polynomials  $\nu$ . Expressing  $Z_L$  in terms of  $\lambda_i$  we find a much nicer and uniform expressions for  $Z_L$ :

$$Z_L(z_1, \dots, z_L) = \det_{1 \leq i, j \leq m} (\lambda_{3i-j} - \lambda_{3i+j}), \quad \text{for } L = 2m + 1, \quad (81)$$

$$Z_L(z_1, \dots, z_L) = \det_{1 \leq i, j \leq m-1} (\lambda_{3i-j} - \lambda_{3i+j}), \quad \text{for } L = 2m. \quad (82)$$

This again can be proven using the appropriate row-column manipulations.

## 5 Discussions

In the dense loop model at  $n = 1$  [9] the sum rules of the ground state are nothing but Schur functions and symplectic Schur functions (rather their combination) for periodic and open boundary conditions, respectively. In our case Di Francesco found an expression for the periodic sum rule which is a skew Schur function (35). The expression for the open boundary sum rule (52) or (56) reminds us the symplectic version of the skew Schur function

(35) written in the form similar to the dual Jacobi-Trudi identity (JT). One can consult, for instance, the paper [10], where many JT and dual JT identities were derived for the classical symmetric functions using the Gessel-Viennot algorithm [13].

What is interesting about the formulae (62), (63) is that they remind us the JT identities but with different symmetric polynomials. Presumably, there exists a version of Gessel-Viennot method that defines some symmetric functions by a JT-like identity, one of this symmetric functions will be equal to the  $Z_L^p$ . Similarly, the skew versions of (62), (63) appear for the open boundary sum rule (77), (78), and as well (81), (82).

We would like to emphasize two things. First, we used the prefactors in the recurrence relations: (33), (57) and (72), to generate symmetric polynomials. These polynomials allow to express the solutions of the corresponding recurrence relations in nice determinant forms. Instead of looking for the solutions in terms of Schur functions, or other symmetric functions which form a basis for the space of symmetric functions, one must look for the basis appropriate for the problem in question.

The second point is regarding the solutions of the recurrence relations for open boundary conditions. Given a determinantal expression for the periodic boundary sum rule we assumed the second half of the variables  $z_i$  ( $i > m$ ) in the list  $z_1, \dots, z_{2m}$  to be the inverses of those in the first half ( $z_i$ ,  $i \leq m$ ) and then observed that the resulting matrix possesses a certain symmetry. Using the appropriate transformation (45) for this matrix turned it into a block diagonal form, which means that its determinant is a product of the determinants of the blocks. The blocks of this matrix turn out to be proportional to the solution for the open boundary sum rules ((47), (48) and, in the case of the second recurrence relation, the numerator of (78)). This approach works well for both recurrence relations considered here.

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